Section 2.3: Acceleration-velocity models

We have already done constant acceleration.
We now let the acceleration vary.
Most problems in the book do acceleration due to gravity, modified by some resistance.

We study:

- Resistance proportional to speed:
  dv/dt = -g - pv
- resistance proportional to v^2
- dv/dt = -g - pv|v|
- Force that depends on position. e.g. gravity at a distance from our planet.

$$\frac{d^2r}{dt^2} = \frac{-GM}{r^2}$$

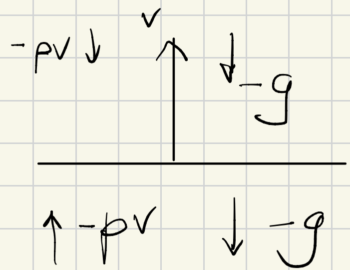
New vocabulary: Terminal velocity, escape velocity.

Resistance proportional to speed, such as

$$\frac{dv}{dt} = -g - pv$$

Here v = dx/dt where x = distance measured vertically up, g is the acceleration due to gravity.

p is a resistance coefficient.



Solution.

$$\int \frac{dv}{-g-pv} = \int dt$$

$$-\frac{1}{p}\ln(g+pv) = t + C, \quad \ln(g+pv) = -pt+B$$

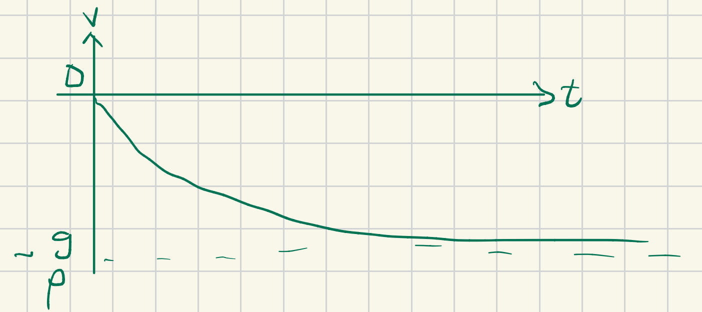
$$g+pv = e^{-pt+B} = Ae^{-pt}$$

$$v = De^{-pt} - \frac{g}{p}$$

If $v(0) = 0$ then $0 = D - \frac{g}{p}$

$$D = \frac{g}{p}$$

$$v = \frac{g}{p}\left(e^{-pt} - 1\right)$$



$-\frac{g}{p}$ is the terminal velocity.

Pre-class Warm-up!!!

A skydiver falls with air resistance proportional to speed, according to the equation:

$$\frac{dv}{dt} = -g - pv$$

If g = 10 m/s^2 and p = 0.2 s^-1 what is the skydiver's terminal speed?

a. 5 m/s

b. 10 m/s

✓ c. 50 m/s

d. 200 m/s

e. Not enough information to tell.

Put $\frac{dv}{dt} = 0$

$= -g - pv$

$v = -\frac{g}{p} = -\frac{10}{0.2}$

$= -50$

Page 101 question 9

A motor boat weighs 32,000 lb and its motor provides a thrust of 5000lb. Assume that the water resistance is 100 pounds for each foot per second of the speed v of the boat. Then

$$1000\,\frac{dv}{dt} = 5000 - 100v$$

If the boat starts from rest, what is the m animus velocity that it can attain?

How long does it take the boat to attain 90% of its limiting velocity?

$$v = 50\left(1 - e^{-t/10}\right) = 45$$

$$t = 10\ln 10$$

Motion with resistance to motion proportional to v^2

Going up or down with resistance proportional to v and gravity g:

$$\frac{dv}{dt} = -g - \rho v$$

Review

Going up with resistance proportional to v^2, gravity g:

$$\frac{dv}{dt} = -g - \rho v^2$$

Going down with resistance proportional to v^2, gravity g:

$$\frac{dv}{dt} = -g + \rho v^2$$

Solutions:

$$v = De^{-\rho t} - \frac{g}{\rho}$$

$$v = \sqrt{\frac{g}{\rho}} \tan\left(D - t\sqrt{\rho g}\right) \qquad (13)$$

$$v = \sqrt{\frac{g}{\rho}} \tanh\left(D - t\sqrt{\rho g}\right) \qquad (16)$$

How to solve $\frac{dv}{dt} = -g - \rho v^2$

$$\int \frac{dv}{g+\rho v^2} = \int -dt = \int \frac{dv}{g\left(1+\frac{\rho}{g}v^2\right)} = \frac{1}{g}\sqrt{\frac{g}{\rho}} \tan^{-1}\left(\sqrt{\frac{\rho}{g}}\, v\right) = \frac{1}{\sqrt{\rho g}} \tan^{-1}\left(\sqrt{\frac{\rho}{g}}\, v\right) = -t + C$$

$$\sqrt{\frac{\rho}{g}}\, v = \tan\left(C\sqrt{\rho g} - t\sqrt{\rho g}\right), \quad v = \sqrt{\frac{g}{\rho}} \tan\left(D - t\sqrt{\rho g}\right), \quad D = C\sqrt{\rho g}.$$

Page 102 question 17 (like question 20).

A bolt is shot straight up from the ground (y = 0) at time t = 0 with initial velocity v_0 = 49 m/s. Take g = 10 m/s^2 and p = 0.0011. Use equations 13 and 14 to show that the bolt reaches its maximum height of about 108.47 m in about 4.61 s.

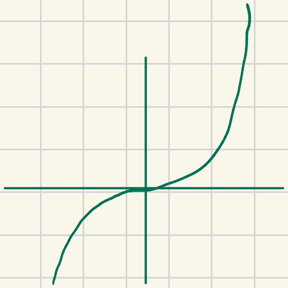
Solution Put $t=0$

$$v_0 = 49 = \sqrt{\frac{g}{p}} \tan(D - 0),$$

$$D = \tan^{-1} 49\sqrt{\frac{0.0011}{10}} = \tan^{-1} \text{some number.}$$

The maximum height happens when $v=0$, so $\tan(D - t\sqrt{pg}) = 0$

$$D = t\sqrt{pg} \qquad t = \frac{D}{\sqrt{pg}} = \text{some number.}$$



Question. The function tan eventually tends to infinity as t increases, suggesting that the bolt goes faster and faster, without bound. What is going on here?

a. This is correct. It is what happens.

b. There is something wrong with the equation we are using.

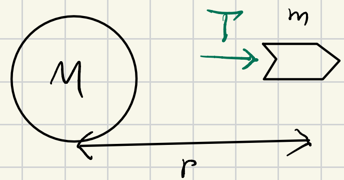
c. There was a mistake in the way the solution was found.

d. Something else.

When the bolt starts going down we use the 'down' equation

$$(13) \quad v = \sqrt{\frac{g}{p}} \tan\left(D - t\sqrt{pg}\right)$$

Newton's law of gravitation.



Newton: $m\frac{d^2r}{dt^2} = -\frac{GMm}{r^2}$

Newton with thrust:

$$m\frac{d^2r}{dt^2} = \frac{-GMm}{r^2} + mT$$

We solve: $\frac{d^2r}{dt^2} = -\frac{GM}{r^2} + T$

Reduce the order: $v = \frac{dr}{dt}$, $\frac{d^2r}{dt^2} = \frac{dv}{dt}$

$$= \frac{dv}{dr}\frac{dr}{dt} = v\frac{dv}{dr} = -\frac{GM}{r^2} + T$$

$$\int v\,dv = \int\left(-\frac{GM}{r^2} + T\right)dr$$

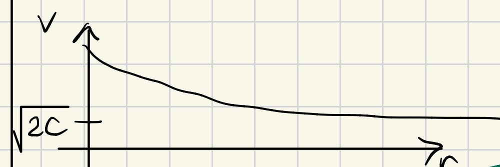
$$\frac{v^2}{2} = \frac{GM}{r} + Tr + C$$

$$v = \sqrt{\frac{2GM}{r} + 2Tr + 2C}$$

In questions T never appears. It does appear in an example in the text with a moon lander. Otherwise forget T.

$$v = \sqrt{\frac{2GM}{r} + 2C} \qquad C = \frac{v_0^2}{2} - \frac{GM}{r_0} > 0$$

if $v_0 > \sqrt{\frac{2GM}{r_0}}$



The escape velocity is $\sqrt{\frac{2GM}{r_0}}$

If a ball is thrown from the surface of the earth and r_0 = radius of the earth, the ball never comes down.

Escape velocity: $\sqrt{\frac{2GM}{R}}$

The following values can be found in places in the book, but it is not easy to find them:

$G = 6.6726 \times 10^{-11}\ N(m/kg)^2$ on p99

$M = 5.975 \times 10^{24}$ kg on p100

$R = 6.378 \times 10^{6}$ m on p100
(radius of the earth)

The moon has

$M = 7.35 \times 10^{22}$ kg p99

Page 102 question 24(a)

To what radius must the earth be compressed to be a black hole (meaning the escape velocity is c = 3 x 10^8 m/s) ?

Solution.

Find R so that $\sqrt{\frac{2GM}{R}} = c$

$$R = \frac{2GM}{c^2} \approx \frac{2 \cdot 6 \cdot 10^{-11} \cdot 6 \cdot 10^{24}}{(3 \times 10^8)^2}$$

$$\approx 8 \times 10^{-3} \text{m} = 0.8 \text{cm}$$

Like page 102 question 25 (on the HW)

A projectile is launched straight up from the earth's surface. What must the initial velocity be for it to reach a height of 10km ?

$$v = \sqrt{\frac{2GM}{r} + 2C}$$

$G = 6.6726 \times 10^{-11}\ N(m/kg)^2$ on p 99

$M = 5.975 \times 10^{24}\ kg$ on p100

$R = 6.378 \times 10^{6}\ m$ on p100

(radius of the earth)